SRI AKILANDESWARI WOMEN'S COLLEGE ,WANDIWASH

# SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS IN REAL LIFE

CLASS : III UG MATHEMATICS

Ms.S.DEEPIKA
Assistant professor
Department of mathematics
SWAMY ABEDHANANDHA EDUCATIONAL TRUST,WANDIWASH

# SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS IN REAL LIFE

# CONTENT

1. Introduction
2. Preliminaries
3. Variable  Separable method and Substitution method
4. Homogeneous Form and Integrating Factor method
5. Real life applications

# ABSTRACT

This project deals with Solutions of Ordinary Differential Equations in real life. Ordinary Differential Equation is a very important mathematical tool applied in many years of Engineering and Science. The paper will talk about the methods and solution of Ordinary differential Equations and the applications of Ordinary Differential Equations in real life. The applications of Ordinary Differential Equations in various fields like Newton’s Law of Cooling/Warming, Population Growth and Radioactive Decay.

# INTRODUCTION

'Differential Equation' began with Leibniz, the Bernoulli brothers and others from the 1680s not long after Newton's 'fluxional equation' in the 1670s. The name "differential equations" itself suggests that these are equations where in the unknowns are connected through the concept of derivatives. The latent significance phenomena become apparent. This branch of mathematics called "differential equations" is like bridge linking mathematics and science with its applications. Hence, it is rightly considered as the language of the

Thus an equation involving ordinary derivatives of an unknown function is called an "ordinary differential equation".

In order to solve problems in geometry and physics, Newton and Leibniz created the field of differential equations along with calculus. It was crucial in the Bernoulli family, Euler, and others developing Newtonian physics. Some of the applications of differential equations in our daily life are found in mobile phones, motor cars, air flights, weather forecast, internet, health care, or in many other daily activities.

# PRELIMINARIES

## Differential Equation:

In Mathematics, a **differential equation** is an equation that contains one or more functions with its derivatives.

The derivatives of the function define the Rate of Change of a function at a point. It is mainly used in fields such as Physics, Engineering and Biology, so on. It is denoted as

$$f(\mathbf{x}) = \frac{dy}{dx}$$

Here “x” is an independent variable and “y” is a dependent variable.

**Example**:

$$\frac{dy}{dx} = \sin x$$

**Order of a Differential Equation:**

The **order of a differential equation** is the highest order derivative in the differential equation.

**Example:**

$$\frac{dy}{dx} = x + y + 5$$

The differential equation is of order 1.

**Degree of a Differential Equation:**

If a differential equation is expressible in a polynomial form, then the integral power of a highest order derivative appears is called the Degree of the differential equation.

**Example:**

$$1 + \left(\frac{dy}{dx}\right)^2 = y^2 \left(\frac{d^2y}{dx^2}\right)^2$$

The degree of the differential equation is 2.

**Note:**

Order and degree of a differential equation are always positive integer.

**Types of Differential Equation:**

Differential equations can be divided in several types namely,

- Ordinary Differential Equation
- Partial Differential Equation
- Linear Differential Equation
- Non – Linear Differential Equation
- Homogeneous Differential Equation
- Non - Homogeneous Differential Equation

## ORDINARY DIFFERENTIAL EQUATION:

If a differential equation contains only Ordinary derivatives of one or more functions with respect to a single independent variable. It is said to be an Ordinary Differential Equation (ODE).

## EXAMPLE:

$$\frac{dx}{dt} + \frac{dy}{dt} = 3x - 4y$$

# VARIABLES SEPARABLE METHOD AND SUBSTITUTION METHOD

## VARIABLES SEPARABLE METHOD:

In solving differential equations, separable of variables was introduced initially by Leibniz and later it was formulated by John Bernoulli in the year 1694.

Finding a solution to a first order differential equation will be simple if the variables in the equation can be separated. An equation of the form $f_1(x)g_1(y)dx + f_2(x)g_2(y)dy = 0$ is called an equation with variable separable or separable equation.

Rewrite the given differential equation as,

$$\frac{f_1(\mathrm{x})}{f_2(\mathrm{x})}dx = -\frac{g_2(\mathrm{y})}{g_1(\mathrm{y})}dy.$$

Integration of both sides of the yields, the general solution of given differential equation as $\int \frac{f_1(\mathrm{x})}{f_2(\mathrm{x})}dx = -\int \frac{g_2(\mathrm{y})}{g_1(\mathrm{y})}dy + C.$

where C is an arbitrary constant.

# Example:

Solve $(1+x^2)\frac{dy}{dx} = 1+y^2$

**Solution:**

Given that $(1+x^2)\frac{dy}{dx} = 1+y^2$ ... (1)

The given solution is written in the variables separable form

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2} \quad \text{... (2)}$$

Integrating both sides of (2), we get

$$\tan^{-1} y = \tan^{-1} x + C$$

(3)

$$\tan^{-1} y - \tan^{-1} x = tan^{-1}\left(\frac{y-x}{1+xy}\right) \quad \ldots (4)$$

Using (4) in (3) leads to

$$tan^{-1}\left(\frac{y-x}{1+xy}\right) = C$$

$$\Rightarrow \qquad \frac{y-x}{1+xy} = \tan C = a(\text{say}).$$

$$y - x = a(1 + xy)$$

Thus, $y - x = a(1 + xy)$ gives the required solution.

# SUBSTITUTION METHOD:

Let the differential equation be of the form

$$\frac{dy}{dx} = f(ax + by + c).$$

- If $a \neq 0$ and $b \neq 0$, then the substitution $ax + by + c = z$ reduces the given equation to the variable separable form.
- If $a = 0$ or $b = 0$, then the differential equation is already in separable form.

Substitution method is also known as Reducible variable separable method.

**Example:**

Solve $y' = \sin^2(x - y + 1)$

**Solution:**

Given that $y' = \sin^2(x - y + 1)$

Put $z = x - y + 1$,

Differentiating z with respect to x,

$$\frac{dz}{dx} = 1 - \frac{dy}{dx}$$

$$1 - \frac{dz}{dx} = \sin^2 z$$

i.e.), $$\frac{dz}{dx} = 1 - \sin^2 z$$

$$\frac{dz}{dx} = \cos^2 z$$

Separating the variables, $\frac{dz}{\cos^2 z} = dx$

$$\sec^2 z\, dz = dx$$

On integrating, $\int sec^2 z\, dz = \int dx$

$$\tan z = x + C$$

$$\tan(x - y + 1) = x + C$$

Hence, $\tan(x - y + 1) = x + C$ is the required solution.

# HOMOGENEOUS FORM AND INTEGRATING FACTOR METHOD

## HOMOGENOUS FORM

**HOMOGENEOUS DIFFERENTIAL EQUATION:**

An ordinary differential equation is said to be in homogeneous form, if the differential equation is written as $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$.

If M(x,y)dx + N(x,y)dy = 0 is a homogeneous

equation, then change

of variable y = vx, transforms into a separable equation in the variable v and x .

**Example:**

Solve $(x^2 - 3y^2)dx + 2xydy = 0$

**Solution:**

We know that the given equation is homogeneous.

Now, we rewrite the given equation as $\frac{dy}{dx} = \frac{3y}{2x} - \frac{x}{2y}$ .

Taking $y = vx$, we have $v + x\frac{dv}{dx} = \frac{3v}{2} - \frac{1}{2v}$

$$x\frac{dv}{dx} = \frac{v^2-1}{2v}$$

Separating the variables, we get $\frac{2v\,dv}{v^2-1} = \frac{dx}{x}$ .

On integrating the variable,

$$\log|v^2 - 1| = \log|x| + \log|C|.$$

$|v^2 - 1| = |Cx|$, Where C is an arbitrary constant .

Replace $v$ by $\frac{y}{x}$ to get, $\left|\frac{y^2}{x^2} - 1\right| = |Cx|$

$$|y^2 - x^2| = |Cx^3|$$

$$y^2 - x^2 = \pm Cx^3$$

Therefore, $y^2 - x^2 = kx^3$ gives the general solution

1765).

This method was used to solve linear ordinary differential equation. Let the differential equation be of the form, $\frac{dy}{dx} + Py = Q$ then the solution of this equation is,

$$ye^{\int Pdx} = \int Q\, e^{\int Pdx} dx + C$$

Here, $e^{\int Pdx}$ is known as the **integrating factor (I.F).**

**Example:**

Solve $\frac{dy}{dx} + 2y = e^{-x}$

**Solution:**

Given that $\frac{dy}{dx} + 2y = e^{-x}$ ... (1)

Here $\mathrm{P} = 2$ ; $Q = e^{-x}$

$$\int Pdx = \int 2dx = 2x$$

$$\mathrm{I.F.} = e^{\int Pdx} = e^{2x}$$

We know that, $ye^{\int Pdx} = \int \mathrm{Q}\, e^{\int Pdx} dx + C$

$$ye^{2x} = \int e^{-x} e^{2x} dx + C$$

Hence, $ye^{2x} = e^{-x} + Ce^{-2x}$ is the required solution.

# REAL LIFE APPLICATIONS

## POPULATION GROWTH:

We consider the growth of a population (for example, human, an animal or a bacteria colony) as a function of time t. This population increases exponentially with time. This law of population growth is called Malthusian law.

### 5.1.1 FORMULA:

$$P = P_0 \times e^{rt}$$

Where, P - Total population after time t.

$P_0$ - Starting population.

r - Constant of proportionality

t - Time in hours or years.

**Example:**

200 crabs are placed in a pond that can support a population of 2000 crabs. After one month the population of the crabs had increased to 500. How long will it take for the pond to reach full capacity?

**Solution:**

Given, t = 0 p = 200 ; t = 1m p = 500 ; t =? p = 2000

From, $$\frac{dp}{dt} = \mathrm{k} \times \mathrm{p}$$

$$\frac{1}{P} dp = kdt$$

Integrating on both sides, $\int \frac{1}{p} dp = \mathrm{kdt}$

$$p = e^{kt+C}$$

$$p = e^{kt}.e^{C}$$

Since, $e^{C} = C$ $\qquad P = Ce^{kt}$

… (1)

To find C, substitute t = 0, p = 200 into (1), we have

$$200 = Ce^{0t}$$

C = 200

Equation (1) becomes, $\qquad$ P $= 200e^{kt}$

To find k, substitute t = 1, P = 500 into (2)

$$P = 200e^{kt} \qquad \text{… (3)}$$

$$e^{k} = 2.5$$

$$k = \ln(2.5)$$

$$k = 0.9162$$

Equation (3) Become, $p = 200e^{0.91629t}$

Substitute p = 2000 into (3) , $2000 = 200e^{0.91629t}$

$$e^{0.91629t} = 10$$

$$0.91629t = \ln(10)$$

$$t = \frac{\ln(10)}{0.91629}$$

$$t = 2.513\ months$$

It took 2.513 months for the pond to reach full capacity.

**NEWTON’S LAW OF COOLING/WARMING:**

Newton’s law of cooling was developed by Sir Isaac Newton in 1701. Newton noted that the rate of heat loss of a body is directly proportional to the difference in the temperature between the body and its surrounding.

**FORMULA:**

Newton’s law of cooling formula is expressed by,

$$T(t) = T_s + (T_o - T_s)e^{-kt}$$

Newton’s law of warming formula is expressed by,

$$T(t) = T_s + (T_0 - T_s)e^{kt}$$

Where,

$T_f$/ T(t) - Object’s temperature at time ‘t’

$T_o$ - Initial temperature of body

t - Time

k - Constant of proportionality

**Example:**

The C.B.I, team arrived at the scene of a murder at 9.30pm. At that time, the dead body's temperature was 77.9°F. At 10.30 pm, the dead body's temperature was recorded again and it was 75.6°F. Given the room temperature was 72°F, when was the person murdered? (A normal human body temperature is 98.6°F).

**Solution:**

$T_m$ = 72°C (Room temperature) t = 0 T=77°F

t = 1-hour T=75.6°F t =? T=98.6°F

From $$\frac{dT}{dt} = k(T - T_m)$$

Separating variable t and T,

$$\frac{1}{(T-T_m)}\,dT = k\,dt$$

Integrating on both sides,

$$\int \frac{1}{T-T_m}\,dT = \int k\,dt$$

$$\ln(T - T_m) = k\,t + c$$

Taking Exponential on both sides, $T - T_m = e^{kt+c}$

$$T - T_m = e^{kt}\,e^{c}$$

$$T - T_m = ce^{kt}$$

$$T = 72 + ce^{kt} \qquad \dots (1)$$

To find C, substitute t = 0, T = 77.9°F into (1)

From T = 72 + $ce^{kt}$, since, $e^0 = 1$

$$77.9 = 72 + ce^0$$

$$77.9 - 72 = C$$

$$C = 5.9$$

Put, $C = 5.9$ into (1),

$$T = 72 + 5.9e^{kt} \quad \ldots (2)$$

To find k, substitute $t = 1\text{hr}, T = 75.6°\text{F}$ into (2)

$$75.6 = 72 + 5.9e^{k(1)}$$

$$5.9e^{k} = 75.6 - 72 \quad >> \quad 5.9e^{k} = 3.6$$

$$e^{k} = \frac{3.6}{5.9}$$

$$k = \log\left(\frac{3.6}{5.9}\right)$$

$$k = -0.49401$$

Equation (2) becomes,

$$T = 72 + 5.9e^{-0.49401\,(t)} \quad \ldots (3)$$

Substitute T = 98.6°F into (3)

$$98.6 = 72 + 5.9e^{-0.49401\,(t)}$$

$$98.6 - 72 = 5.9e^{-0.49401\,(t)}$$

$$26.6 = 5.9e^{-0.49401\,(t)}$$

$$e^{-0.49401\,(t)} = \frac{26.6}{5.9}$$

Taking log on both sides, we get

$$-0.49401\mathrm{t} = log\left(\frac{26.6}{5.9}\right)$$

$$\mathrm{t} = -\ 3.04843 \text{ hours}$$

Thus, the murder occurred 3.04843 hours before 9.30 pm,

3.04843 hours = 3 hrs 3 minutes. So, the murder occurred at about 6.27pm

## RADIOACTIVE DECAY:

The nucleus of an atom consists of combinations of protons and neutrons. Many of these combinations of protons and neutrons are unstable, that is the atoms decay or transmute into the atoms of another substance. Such nuclei are said to be radioactive.

**FORMULA:**

$$A = A_0 e^{-kt}$$

Where,

A - Number of atoms present after the time t

$A_0$- Number of atoms present at the time t=0

k - Constant of proportionality

t - time

**Example:**

A radioactive isotope has an initial mass 200mg, which two years later is 150mg. Find the expression for the amount of the isotope at any time. What is its half-life?

**Solution:**

Let A be the mass of the isotope remaining after t years, and let –k be the constant of proportionality, where k > 0.

The rate of decomposition is modelled by

$$\frac{dA}{dt} = -kA$$

where the minus sign indicates that the mass is decreasing. It is a separable equation.

Separating the variables, $\frac{dA}{A} = -kdt$

Integrating on both sides, $\int \frac{dA}{A} = \int -kdt$

$$\log|A| = -kt + \log|C| \text{ (or)}$$

$$A = Ce^{-kt}.$$

... (1)

Given that the initial mass is 200mg.

That is, $A = 200$ when $t = 0$. Substituting that in equation (1),

$$200 = Ce^{-k(0)}$$

Since, $e^0 = 1$, $\quad 200 = C(1)$

$$C = 200$$

Substituting, $C = 200$ in equation (1), we get,

$$A = 200e^{-kt} \quad \text{... (2)}$$

Also, $A = 150$ when $t = 2$, then equation (2) becomes,

$$150 = 200e^{-k(2)}$$

$$3 = 4e^{-k(2)}$$

$$\frac{3}{4} = e^{-k(2)}$$

Taking log on both sides, we get : $\quad k = \frac{1}{2}\log\left(\frac{4}{3}\right)$

Equation (2) becomes, $A(t) = 200e^{-\frac{t}{2}\log\left(\frac{4}{3}\right)}$

Hence, $A(t) = 200e^{-\frac{t}{2}\log\left(\frac{4}{3}\right)}$ is the mass of isotope remaining after t years.

The half-life $t_h$ is the time corresponding to A = 100mg.

$$100 = 200e^{-\frac{t_h}{2}\log\left(\frac{4}{3}\right)}$$

$$\frac{1}{2} = e^{-\frac{t_h}{2}\log\left(\frac{4}{3}\right)}$$

Taking log on both sides, we get

$$\log\left(\frac{1}{2}\right) = -\frac{t_h}{2}\log\left(\frac{4}{3}\right)$$

$$\log\left(\frac{1}{2}\right) = \frac{t_h}{2}\log\left(\frac{3}{4}\right)$$

$$\text{Half-life } t_h = \frac{2\log\left(\frac{1}{2}\right)}{\log\left(\frac{3}{4}\right)}.$$

Hence, this is the required solution.

# CONCLUSION

Ordinary differential equation is a powerful tool using in different areas of maths, physics and engineering with the case of application in research software made. It is possible to stimulate the ordinary derivative directly which has made a good advancement in the research field. The methods of solving ordinary differential equation using variables separable method, substitution method, homogeneous form and integrating factor method are discussed here. Also we discussed some real life applications of first order deferential equations like Newton cooling or warming, population growth and radioactive decay. In future, I want to explore more things about ordinary differential equations.

# REFERENCES

[1] Arnold, V. Ordinary Differential Equation, Springer-Verlag, Berlin Heidelberg, 1992.

[2] Brauer, F., and J. Nohel. Ordinary Differential Equation (2nd Edition), Benjamin, New York, 1973.

[3] Braun, M. Differential Equation and Their Application: An Introduction to Applied Mathematics (3rd Edition), Springer-Verlag, New York, 1983.

[4] Campbell, S.L. An Introduction to Differential Equations and Their Applications. Wadsworth, Belmont, 1990.

[5] Coddindton, E., and N. Levinson. Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[6] Corduneanu, C. Principles of Differential and Integral Equations, Allyn & Bacon, Boston, 1971.

[7] Derrick, W., and S. Grossman. Elementary Differential Equations with Applications (2nd Edition), Addison-Wesley, Reading, Mass., 1976.

[8] Finizio, N., and G. Ladas. Ordinary Differential Equations with Modern Applications (2nd Edition), Wadsworth, Belmont, Calif., 1982.

[9] Hale, J. Ordinary Differential Equations, Wiley-Inter science, New York, 1969.

[10] Hartman, P. Ordinary Differential Equations, Wiley, New York, 1964.

[11] Leighton, W. An Introduction to the Theory of Ordinary Differential Equations, Wadsworth, Inc. Bellmont, California, 1976.

[12] Miller, R., and A. Michel. Ordinary Differential Equations, Academic Press, New York, 1982.

[13] Reiss, E., A. Callegri, and D. Ahluwalia. Ordinary Differential Equations with Applications, Holt, Rinehart, & Winston, New York, 1976.

[14] Sanchez, D., R. Allen, and W. Kyner. Differential Equations, An Introduction, Addison-Wesley, Reading, Mass., 1983.

[15] Shepley L.Ross, Introduction to Ordinary Differential Equations (4th Edition).

[16] Wylie, C. R. Differential Equations, McGraw-Hill, New York, 1979.

# THANK YOU